六 Laurent Expansion

1. Two-side Power Series

In topic 4, we discussed the power series

$$a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

and we gave out the convergence radius $|z-z_0|<R$

Now, let's consider another form of series

$$\frac{a_{-1}}{z-z_0} + \frac{a_{-2}}{(z-z_0)^2} + \cdots$$

This is also a power series, but the indexes are negative.
If we use ξ to substitute $(z-z_0)^{-1}$, we have the form

$$a_{-1}\xi + a_{-2}\xi^2 + \cdots$$

So, we can have another convergence radius $|\xi| < \frac{1}{r}$,
or $|z-z_0|>r$

If we combine the two power series

$$\sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$$

But pay attention, "$R \geq r$", i.e "$r < |z-a| < R$", or the combined power series won't converge, or won't correspond to an analytic function $f(z)$.

also, it can be checked that we can take n-th order derivative.

2. Laurent Expansion.

A analytic function $f(z)$ converging in $r<|z-z_0|<R$ can be expanded uniquely as.

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$$

where

$$a_n = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\,dz \qquad n=0, \pm 1, \pm 2, \cdots$$

γ is the closed path in $r<|z-z_0|<R$.

We can see that, Laurent expansion is a generalized version of Taylor expansion.

Eg. Compute the Laurent expansion of

$$f(z) = \frac{1}{(z-1)(z-2)}$$

Because $f(z)$ has two singularities $z=1$ & $z=2$, so let's divide the whole space to

① $|z|<1$

② $1<|z|<2$

③ $2<|z|<\infty$

Solution:

$$f(z) = \frac{1}{1-z} - \frac{1}{2-z}$$

① for $|z|<1$

$$f(z) = \frac{1}{1-z} - \frac{1}{2(1-\frac{z}{2})}$$

$\because |\frac{z}{2}|<1 \quad \therefore |\frac{z}{2}|$ is also in the convergence region.

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

$$\frac{1}{1-\frac{z}{2}} = \sum_{n=0}^{\infty} (\frac{z}{2})^n$$

$$\therefore f(z) = \sum_{n=0}^{\infty} z^n - \frac{1}{2}\sum_{n=0}^{\infty} (\frac{1}{2})^n z^n$$

$$= \sum_{n=0}^{\infty} (1-\frac{1}{2^{n+1}}) z^n$$

② for $1<|z|<2$.

although $|z|>1$, but $|\frac{1}{z}|<1$, so we can make use of this

and $|z|<2 \Rightarrow |\frac{z}{2}|<1$

$$f(z) = \frac{1}{1-z} - \frac{1}{2-z}$$

$$= -\frac{1}{z}\frac{1}{1-\frac{1}{z}} - \frac{1}{2(1-\frac{z}{2})}$$

$$= -\frac{1}{z}\cdot\sum_{n=0}^{\infty}\frac{1}{z^n} - \frac{1}{2}\sum_{n=0}^{\infty}\frac{z^n}{2^n}$$

(3) When $2<|z|<+\infty$

Now, $|\frac{1}{z}|<1$, $|\frac{2}{z}|<1$

$$f(z)=\frac{1}{z}\cdot\frac{1}{1-\frac{2}{z}}-\frac{1}{z}\cdot\frac{1}{1-\frac{1}{z}}$$

$$=\frac{1}{z}\sum_{n=0}^{\infty}\frac{2^n}{z^n}-\frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n}$$

$$=\sum_{n=2}^{\infty}\frac{2^{n-1}-1}{z^n}$$

3. Three types of isolated singularities

If a is a singularity of $f(z)$, then, we can use Laurent expansion to expand $f(z)$ in a non-central neighbourhood of a.

$$f(z)=\sum_{n=-\infty}^{\infty}a_n(z-a)^n$$

and $\sum_{n=-\infty}^{-1}a_n(z-a)^n$ ----- principal part

$\sum_{n=0}^{\infty}a_n(z-a)^n$ ----- regular part

① removable singularity

When the principle part is zero, this singularity is removable.

② pole of order k

if we can find an integer $k>0$ s.t

$C_n=0$ for $n<-k$ and $C_k\neq 0$,

E.g. we have

$$\frac{C_{-k}}{(z-a)^k} + \frac{C_{-k+1}}{(z-a)^{k-1}} + \cdots \frac{C_{-1}}{(z-a)}$$

and $C_{-k-1}, C_{-k-2}, \cdots$ are all zeros.

③ essential singularity
if it has infinite principle parts.

Now let's analyze all these three types of singularities.

① Removable Singularity

a is a removable singularity

$\Leftrightarrow$ $f(z)$'s principle part at a is zero (definition)

$\Leftrightarrow$ $\lim_{z \to a} f(z) = b \neq \infty$ $\quad$ $(\lim_{z\to a} f(z) = \lim_{z \to a} a_0 + a_1(z-a) + \cdots$
$= a_0 \neq \infty)$

$\Leftrightarrow$ $f(z)$ is bounded in certain non-central neighbourhood of a.

$\because$ if $\lim_{z\to a} f(z) = b$. $\Leftrightarrow$ $\forall \varepsilon > 0$, we can have $\delta > 0$ s.t

$$|f(z) - b| < \varepsilon \text{ if } |z-a| < \delta$$

So, $|f(z) - b| < \varepsilon \Rightarrow |f(z)| - |b| < \varepsilon \Leftrightarrow |f(z)| < \varepsilon + |b|$

so it's bounded

② pole of k-th order

a is a k-th order pole

<=> the principal part of $f(z)$ at a is

$$\frac{C_{-k}}{(z-a)^k} + \frac{C_{-k+1}}{(z-a)^{k-1}} + \cdots + \frac{C_{-1}}{(z-a)}, \quad C_{-k} \neq 0 \qquad \text{(definition)}$$

<=> $f(z)$ can be represented as

$$f(z) = \frac{1}{(z-a)^k} \cdot \lambda(z)$$

where $\lambda(z)$ is analytic a $N_\varepsilon(a)$, $\lambda(a) \neq 0$

<=> $g(z) = \frac{1}{f(z)}$ has a k-th order zero point at a

③ essential singularity

a is an essential singularity

<=> $\lim_{z \to a} f(z)$ doesn't exist

i.e. $\lim_{z \to a} f(z) \neq \infty$ and $\lim_{z \to a} f(z) \neq b < \infty$